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Solution of a Circular Ring Structural Problem

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ABSTRACT

Formulae for calculating the vertical shear, bending moment, torsion, and deflection at midspan are presented for the case of an endless circular ring of constant cross section which is held by N frictionless supports and symmetrically loaded by N concentrated forces and/or a uniformly distributed load, both of which act normal to the plane of the ring.

I. SOLUTION OF A CIRCULAR RING STRUCTURAL PROBLEM

An endless circular ring of constant cross section is supported on N equally spaced and frictionless supports, and is loaded normal to the plane of the ring by a uniformly distributed load of total magnitude P and/or by N concentrated loads, each of magnitude F/N acting midway between the supports. See Fig. 1 for a typical example. If it is assumed that the radius R of the ring is large in comparison to its cross-sectional dimensions, then the following expressions obtain for the vertical shear, moment, torsion, and deflection at midspan.

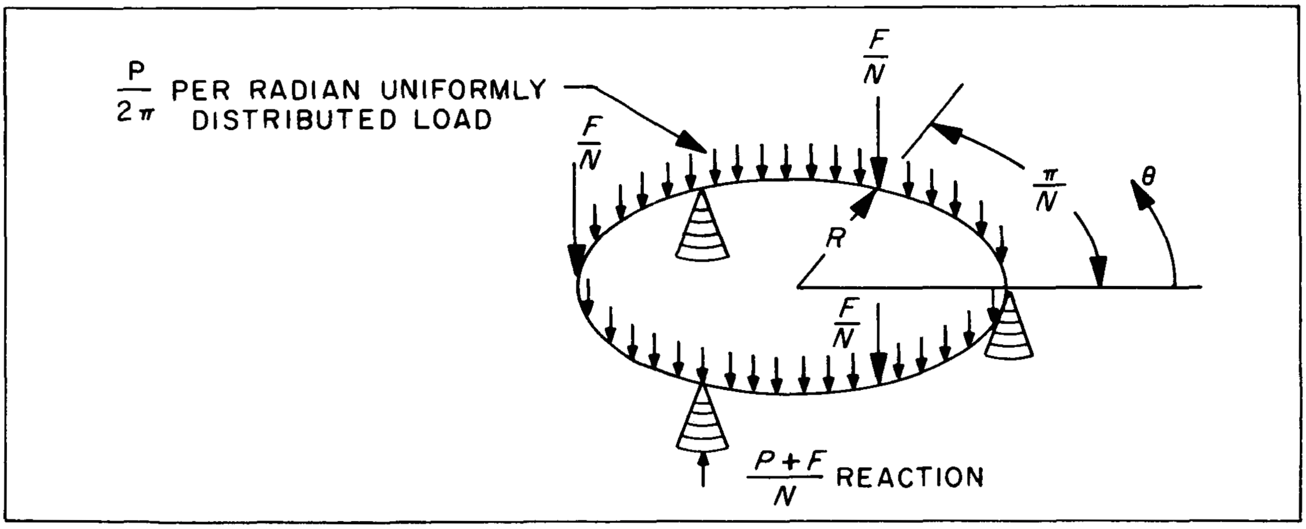

Fig. 1. Load and support configuration

Vertical Shear

$$S = \frac{P + F}{2N} - \frac{P\theta}{2\pi} \tag{1}$$

Bending Moment

$$M = \frac{FR}{2N}\left[\sin\theta - \left(\csc\frac{\pi}{N} - \cot\frac{\pi}{N}\right)\cos\theta\right] + \frac{PR}{2N}\left[-\frac{N}{\pi} + \sin\theta + \cot\frac{\pi}{N}\cos\theta\right] \tag{2}$$

Torsion

$$T = \frac{FR}{2N}\left[1 - \cos\theta - \left(\csc\frac{\pi}{N} - \cot\frac{\pi}{N}\right)\sin\theta\right] + \frac{PR}{2N}\left[1 - \frac{N\theta}{\pi} - \cos\theta + \cot\frac{\pi}{N}\sin\theta\right] \quad (3)$$

Equations (1), (2), and (3) apply to the range $0 \leqq \theta \leqq \pi/N$. The moment M is positive when compression is produced in the upper fibers. The torsion T is positive when clockwise as viewed by looking in the positive θ direction. The moment and torsion distributions, produced by P and F, are shown qualitatively in Fig. 2.

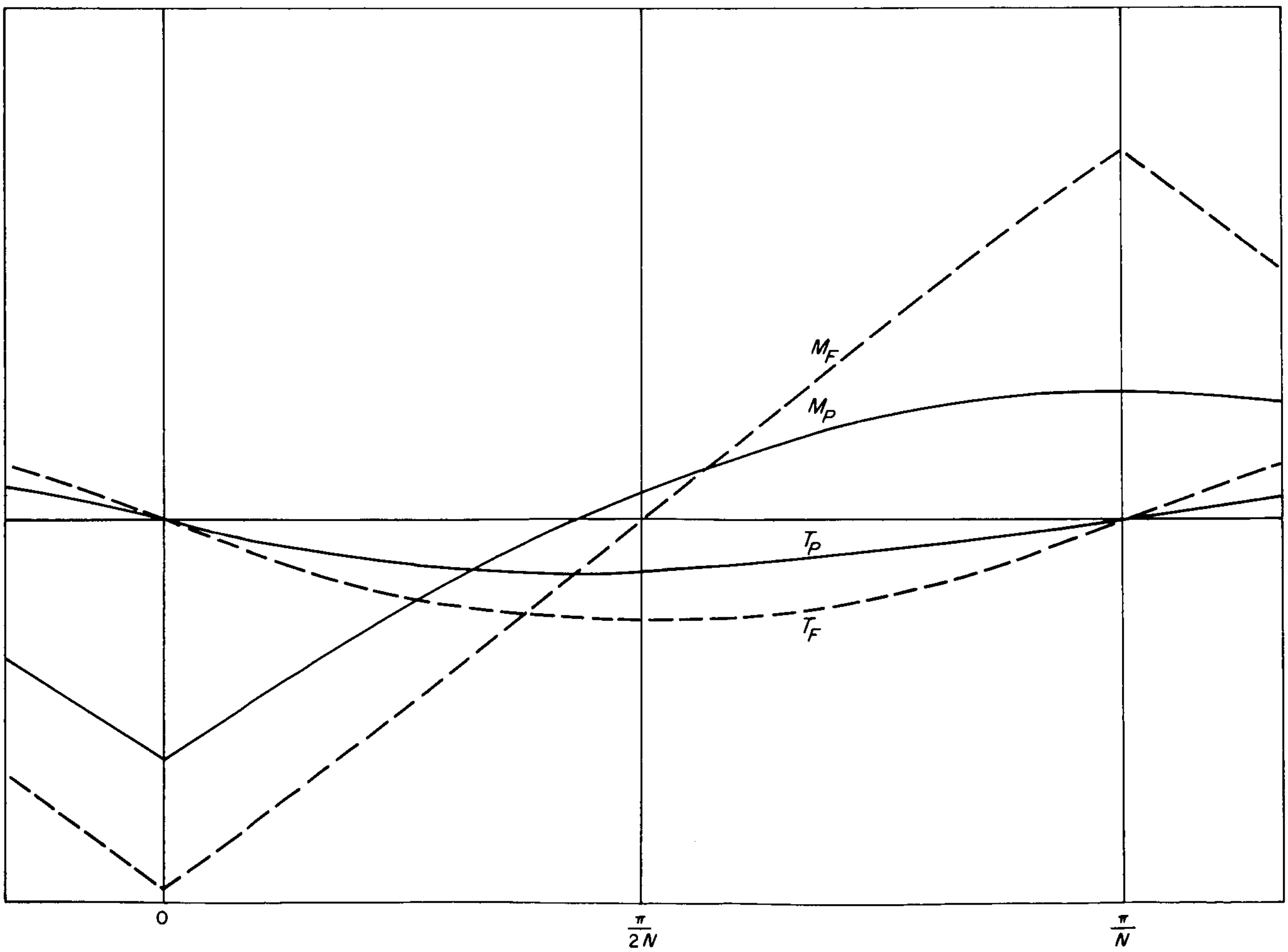

Fig. 2. Moment and torsion distribution

The deflection at midspan δ (positive downward) is:

$$\delta = \frac{FR^3}{2N^2\,EI}\left[\frac{\left(\pi - N\sin\frac{\pi}{N}\right)\left(1-\cos\frac{\pi}{N}\right)}{\sin^2\frac{\pi}{N}}\right]$$

$$+\frac{FR^3}{2N^2\,GK}\left[\frac{\pi\left(1-\cos\frac{\pi}{N}+\sin^2\frac{\pi}{N}\right)-3N\left(1-\cos\frac{\pi}{N}\right)\sin\frac{\pi}{N}}{\sin^2\frac{\pi}{N}}\right]$$

$$+\frac{PR^3}{4N^2\,EI}\left[\frac{\left(\pi - N\sin\frac{\pi}{N}\right)\left(1-\cos\frac{\pi}{N}\right)}{\sin^2\frac{\pi}{N}}\right]$$

$$+\frac{PR^3}{4N^2\,GK}\left[\frac{\pi\left(1-\cos\frac{\pi}{N}+\sin^2\frac{\pi}{N}\right)-3N\left(1-\cos\frac{\pi}{N}\right)\sin\frac{\pi}{N}}{\sin^2\frac{\pi}{N}}\right] \tag{4}$$

where E is the modulus of elasticity, G is the shear modulus, I is the moment of inertia of area of the cross section (about the horizontal axis), and K is the torsional deflection factor.

Equation (4) may be written as follows, where the a and b values are listed in Table 1 for various values of N:

$$\delta = \frac{R^3}{N^4}\left\{F\left[\frac{a}{2EI}+\frac{b}{2GK}\right]+P\left[\frac{a}{4EI}+\frac{b}{4GK}\right]\right\} \tag{5}$$

The maximum negative moment occurs at a support, and is:

$$M_{\substack{max \\ neg}} = -\frac{FR}{2N}\left(\csc\frac{\pi}{N} - \cot\frac{\pi}{N}\right) - \frac{PR}{2N}\left(\frac{N}{\pi} - \cot\frac{\pi}{N}\right) = -\frac{FR}{4N^2}(\eta) - \frac{PR}{6N^2}(\lambda) \qquad (6)$$

The maximum positive moment occurs at midspan, and is:

$$M_{\substack{max \\ pos}} = \frac{FR}{2N}\left(\csc\frac{\pi}{N} - \cot\frac{\pi}{N}\right) + \frac{PR}{2N}\left(-\frac{N}{\pi} + \csc\frac{\pi}{N}\right) = \frac{FR}{4N^2}(\eta) + \frac{PR}{12N^2}(\omega) \qquad (7)$$

The maximum torsion caused by F occurs at the quarter-span point. The maximum torsion caused by P occurs at approximately the 0.21 span point; however, as may be seen from Fig. 2, it is only slightly greater than the torsion at the quarter-span point. The torsion at the quarter-span point, which is very near to the maximum, is:

$$T_{\substack{quarter \\ span}} = \frac{R}{2N}\left(F + \frac{P}{2}\right)\left(1 - \sec\frac{\pi}{2N}\right) = -\frac{R}{2}\left(F + \frac{P}{2}\right)(t) \qquad (8)$$

The η, λ, ω, and t values of Eq. (6), (7), and (8) are listed in Table 1 for various values of N.

If for small values of π/N, the approximations

$$\sin\frac{\pi}{N} \approx \frac{\pi}{N} - \frac{\pi^3}{6N^3}$$

and

$$\cos\frac{\pi}{N} \approx 1 - \frac{\pi^2}{2N^2}$$

are substituted into Eq. (2), (3), and (4), there are obtained respectively:

$$M = \frac{FR}{2N}\left[-\frac{\pi}{2N} + \theta\right] + \frac{PR}{2N}\left[-\frac{\pi}{3N} + \theta - \frac{N\theta^2}{2\pi}\right], \quad 0 \leqq \theta \leqq \frac{\pi}{N} \tag{2a}$$

$$T = 0 \tag{3a}$$

$$\delta = \frac{R^3}{N^4}\left[\frac{F}{2EI}\,\frac{\pi^3}{12} + \frac{P}{4EI}\,\frac{\pi^3}{12}\right] \tag{4a}$$

Equations (2a) and (4a) are precisely the moment and maximum deflection equations of a straight beam of length $2\pi R/N$ which is fixed at both ends and loaded by a uniformly distributed load of $P/2\pi R$ and by a center concentrated load of F/N. Thus the limiting value of a is $\pi^3/12$. By establishing the maximum negative and positive values of Eq. (2a) it is seen that π is the limiting value of η, λ, and ω. Obviously zero is the limiting value of b and t. Therefore the values of Table 1 are compared with the corresponding values of an analogous straight beam.

It should be observed that the results of this analysis cannot be applied to the case of a partial ring with the ends fixed both torsionally and flexurally. In such a case the end torque, that is, the torque at $\theta = 0$ is not zero, as it is for the complete ring.

II. DERIVATION OF THE EQUATIONS

The forces, moments, and torques acting on one free body span are shown in Fig. 3. Because of symmetry $|M_1| = |M_0|$ and $|T_1| = |T_0|$

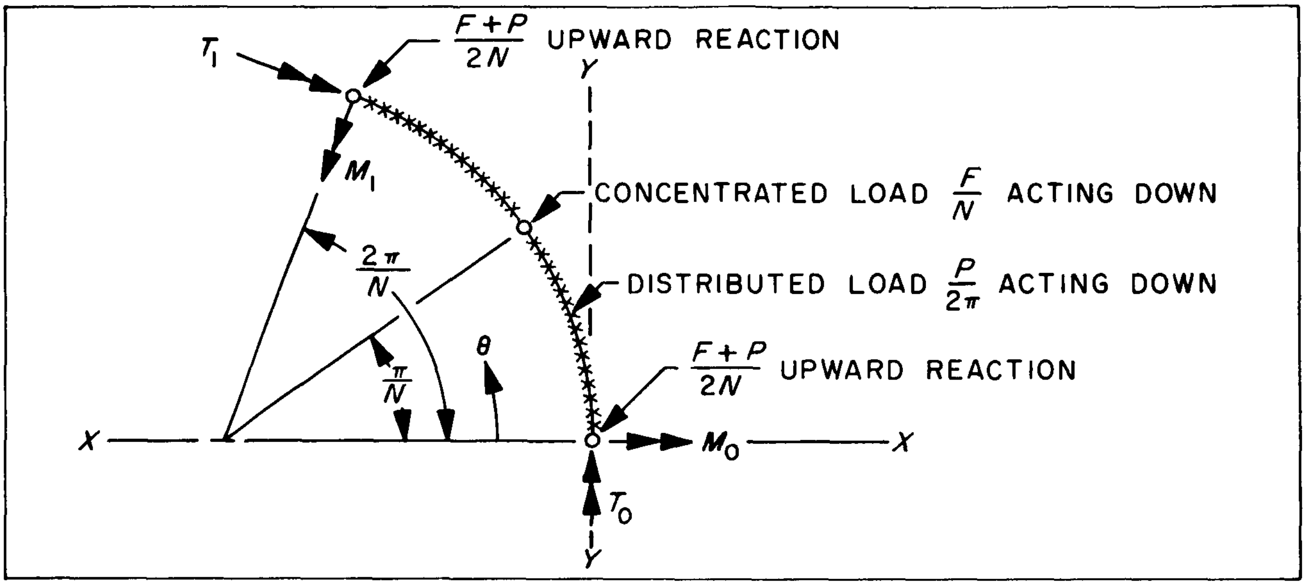

Fig. 3. One free-body span

By summing moments about axes x and y two equations may be written in M_0 and T_0, the simultaneous solution of which yields:

$$M_0 = \frac{FR}{2N}\left(\csc\frac{\pi}{N} - \cot\frac{\pi}{N}\right) + \frac{PR}{2N}\left(\frac{N}{\pi} - \cot\frac{\pi}{N}\right) \tag{9}$$

$$T_0 = 0 \tag{10}$$

Knowing M_0 and T_0 it is easy to write the general expressions for M and T for the range $0 \leqq \theta \leqq \pi/N$. Hence Eq. (2) and (3) are obtained.

Now consider the strain energy, V, in one span.

$$V = 2\left\{\int_0^{\pi/N} \frac{M^2}{2EI} R\,d\theta + \int_0^{\pi/N} \frac{T^2}{2GK} R\,d\theta\right\} \tag{11}$$

The deflection at the midpoint of a span, in the direction of F/N is, by Castigliano's Theorem:

$$\delta = \frac{\partial V}{\partial \frac{F}{N}} = 2R\left\{\frac{1}{EI}\int_0^{\pi/N} M \frac{\partial M}{\partial \frac{F}{N}} d\theta + \frac{1}{GK}\int_0^{\pi/N} T \frac{\partial T}{\partial \frac{F}{N}} d\theta\right\} \tag{12}$$

Differentiating Eq. (2) and (3) with respect to F/N there results:

$$\frac{\partial M}{\partial \frac{F}{N}} = \frac{R}{2}\left[\left(\cot\frac{\pi}{N} - \csc\frac{\pi}{N}\right)\cos\theta + \sin\theta\right] \tag{13}$$

$$\frac{\partial T}{\partial \frac{F}{N}} = \frac{R}{2}\left[1 - \cos\theta + \left(\cot\frac{\pi}{N} - \csc\frac{\pi}{N}\right)\sin\theta\right] \tag{14}$$

Substituting Eq. (2), (3), (13), and (14) into Eq. (12) and integrating and simplifying, there is obtained Eq. (4).

Table 1
Factors for Eq. (5), (6), (7), and (8)

N	*a*	*b*	η	λ	ω	*t*
2	4.5664	1.1328	4.0000	3.8197	4.3606	0.2071
3	3.2610	0.35892	3.4640	3.3982	3.5964	0.05157
4	2.9351	0.18144	3.3136	3.2789	3.3840	0.02060
5	2.8012	0.11035	3.2492	3.2274	3.2940	0.01031
6	2.7318	0.074794	3.2154	3.2004	3.2454	0.005883
7	2.6910	0.05718	3.1949	3.1839	3.2172	0.003674
8	2.6653	0.03999	3.1824	3.1742	3.1968	0.002450
9	2.6479	0.03174	3.1734	3.1671	3.1860	0.001711
10	2.6355	0.02591	3.1677	3.1623	3.1782	0.001250
11	2.6269	0.02149	3.1632	3.1578	3.1739	0.0009345
12	2.6193	0.01709	3.1608	3.1558	3.1680	0.0007166
36	2.5904	0.00307	3.1441	3.1420	3.1514	0.00002646
limit	$\pi^3/12$ (2.5839)	0	π	π	π	0